Philosophia Scientiæ
Travaux d'histoire et de philosophie des sciences

18-3 | 2014
Logic and Philosophy of Science in Nancy (I)

A Scholastic-Realist Modal-Structuralism

Ahti-Veikko Pietarinen

Electronic version
URL: http://journals.openedition.org/philosophiascientiae/978
DOI: 10.4000/philosophiascientiae.978
ISSN: 1775-4283

Publisher
Éditions Kimé

Printed version
Date of publication: 1 October 2014
Number of pages: 127-138
ISBN: 978-2-84174-689-7
ISSN: 1281-2463

Electronic reference
Ahti-Veikko Pietarinen, « A Scholastic-Realist Modal-Structuralism », *Philosophia Scientiæ* [Online], 18-3 | 2014, Online since 28 November 2014, connection on 03 November 2020. URL : http://journals.openedition.org/philosophiascientiae/978 ; DOI : https://doi.org/10.4000/philosophiascientiae.978

A Scholastic-Realist Modal-Structuralism

Ahti-Veikko Pietarinen[1]
Tallinn University of Technology (Estonia)

Résumé : Comment comprendre le discours sur les propriétés de structures, dont l'existence dépend de ce que l'on suppose la réalité de ces structures ? Les mathématiques ne portent pas sur des objets abstraits, pourtant le structuralisme modal respecte la vérité des théorèmes et des preuves, contrairement au fictionalisme. Il est en revanche nominaliste quant aux possibilia. Le problème est que, de peur de réduire les possibilia à des actualités, la logique modale du second ordre qui prétend axiomatiser l'existence modale ne possède pas réellement de sémantique. Il n'existe pas d'identification croisée des entités mathématiques d'ordre supérieur et ainsi nous ne pouvons savoir ce que sont ces entités. Je suggère qu'une notion scolastique de réalisme, émaillé d'identification croisée d'entités d'ordre supérieur, peut nous fournir une sémantique sans s'écrouler. La sémantique des modalités est liée à la logique de Peirce et à sa philosophie pragmaticiste des mathématiques.

Abstract: How are we to understand the talk about properties of structures the existence of which is conditional upon the assumption of the reality of those structures? Mathematics is not about abstract objects, yet unlike fictionalism, modal-structuralism respects the truth of theorems and proofs. But it is nominalistic with respect to possibilia. The problem is that, for fear of reducing possibilia to actualities, the second-order modal logic that claims to axiomatise modal existence has no real semantics. There is no cross-identification of higher-order mathematical entities and thus we cannot know what those entities are. I suggest that a scholastic notion of realism, interspersed with cross-identification of higher-order entities, can deliver the semantics without collapse. This semantics of modalities is related to Peirce's logic and his pragmaticist philosophy of mathematics.

1. Supported by the Estonian Research Council (PUT 267: *The Diagrammatic Mind*, PI A.-V. Pietarinen).

Philosophia Scientiæ, 18(3), 2014, 127–138.

> *Mathematics might be called an art instead of science were it not that the last achievement that it has in view is an achievement of knowing.* [Peirce 1976, Vol. III, 527]

1 Introduction

What is it that mathematicians do and what is the kind of stuff that they are thinking about? The standard method is the axiomatic one. According to it, mathematical (non-logical) axiom systems capture classes of structures as the models of a mathematical system (a theory). Mathematicians then study these structures by applying logical procedures to derive consequences from these axioms. Importantly, it is not predominantly the deductive kinds of logical consequences but the model-theoretic ones that a mathematician is on the lookout whenever new information about the structures is called for.

The axiomatic method, model-theoretically understood, is about logically establishing which structures are possible and which structures are not according to the axioms. The axiomatic method is not about, or at least not predominantly about, what necessarily follows from the axioms. Deductions explain axioms. Mathematicians want to inquire whether something is possible given the non-logical system of axioms. Generally, scientific enigmas are not explained by trying to convince ourselves of why something must be the case but by accounting for how some particular event or phenomenon *could* have happened.

Such features of mathematician's investigative activities are crucial in several respects. First, the subject matter of mathematicians is the study of classes of structures of a certain kind. In this regard, the structuralist approaches that have grown relatively popular of late are not on the wrong track. They nevertheless meet multiple problems, the least of which is not that the axiomatic set theory (say, ZFC that makes first-order models to behave like structures of sets), which is sometimes thought to serve the foundational role as the privileged though inexhaustible structure among all other structures, cannot provide its own model theory and so can hardly serve as a universal medium for the rest of mathematics. In averting the epistemological conundrums that have to do with defining mathematics as the study of abstract objects, by replacing the foundational role of abstract objects with abstract structures, structuralism tends to evocate one level of abstract in lieu of another.

My concern here is not with what kinds of objects structuralists ultimately want mathematical structures to be—the proposals have ranged from the ancient metaphysical particulars to reified patterns and types, both concrete and abstract, including the *ante rem* structures such as number-structures that need no concrete instantiations and express relational features of numbers by providing placeholder structures for mathematical objects such as numbers or sets. Problems with such proposals abound, including *atomicity* and *actuality*. For example, we get no continuous number progression if we prioritize reference to objects with reference to relata as placeholders for objects. And all such relata carry existential assumptions, rendering them structures that *are* somewhere, somehow. Instead, I want to look into the possibility of formulating structuralism in such a wise that is serious about the core feature of the axiomatic method: that the actual mathematical activities have to do with logically establishing which structures are possible according to the axioms.

2 Problems of modal-structuralism

Modal-structuralism (henceforth MS) [Hellman 1989] is a relatively recent addition to the boutique of philosophies of mathematics. It attempts to get away with three major issues, the axiomatic set theory as the foundational basis for all mathematics, as well as atomicity and actualism that characterise versions of commonplace (*ante rem*) structuralisms. MS takes mathematics to be about properties of structures the existence of which is conditional on the assumption of the existence of those structures that they are the properties of. That is, mathematics is about *logically possible structures*. Hellman's proposal also aims at taking into account the actual practices by which mathematics is being done. I argue that it comes close, but not nearly close enough, to the pragmaticist philosophy of mathematics originally proposed by Charles Peirce over a century ago [Pietarinen 2009, 2010].

As to the first, axiomatic set theory, MS relies on category-theoretic morphisms as its first-class citizens. The structures it generates, using a combination of mereology and plural quantification, are point-like constructions. Hence it does resemble set-theoretic constructions in its reliance on morphism as an explication of general properties of mathematical domains such as continuity. In other senses it is also much like set theory, for instance in that it draws the distinction between small and large categories.

Hellman developed the modal version of structuralism in his 1989 book following the suggestions of Putnam [Putnam 1967] who knew about Peirce's philosophy of mathematics [Peirce 2010]. Hellman suggested overcoming set-theoretic commitments by reformulating mathematics in the logic of topos theory. In topos theory, the absolute universe of set theory is replaced with the plurality of the universes of topoi, each providing a possible world in which mathematics is being brewed. The mathematical 'pluriverses' of discourse arguably are then no longer set-theoretic.

I need not enter into the issues as to how MS may avoid the pitfalls of axiomatic set theory, or whether its preferred category-theoretic formulation fares better than some other alternatives, since the success of such formulations is not directly relevant to our philosophical concerns. I rather focus on the logical and metaphysical issues to do with the modal nature of the structures that are emerging here. Since MS takes mathematics to be about properties of structures the existence of which is conditional on the assumption of the existence of those structures that they are the properties of, mathematics in this respect could indeed be taken to be about logically possible structures.

There is nevertheless one remark to be made about particular formulations of MS that is directly relevant to the topic at hand. The reformulation of MS in category theory has one very distinctive feature: it refers to *iconic* forms of structures in reasoning about properties of diagrams. It does so by attributing novel relational features to *diagrammatic representations*, in terms of homomorphism between domains and co-domains analogous to, for instance, continuous maps between topological spaces. This is quite significant, since iconic forms of reasoning, central to mathematical reasoning, are what Peirce proposed over a century ago:

> Diagrammatic reasoning is the only really fertile reasoning. If logicians would only embrace this method, we should no longer see attempts to base their science on the fragile foundations of metaphysics or a psychology not based on logical theory; and there would soon be such an advance in logic that every science would feel the benefit of it. [Peirce 1906, CP 4.571]

Mathematics, in particular, "is observational", he explains,

> in so far as it makes constructions in the imagination according to abstract precepts, and then observes these imaginary objects, finding in them relations of parts not specified in the precept of construction. This is truly observation, yet certainly in a very peculiar sense; and no other kind of observation would at all answer the purpose of mathematics. [Peirce 1902a, CP 1.240]

The relevance of these observations becomes clearer as we proceed.

Let us remark on some of the defining characteristics of modal structuralism. MS strives to dispense with those assumptions of ordinary structuralism that appeal to atomistic postulates about the nature of structures, such as number progressions. It thus dispenses with abstract objects (numbers, sets, etc.), but, unlike, say, fictionalist philosophies of mathematics, respects the truth of mathematical theorems and proofs. The full nominalist sense—that the claims of mathematics are in fact false since what they really are about concerns objects of fiction and not of reality or even imagination—would be a substantial foundationalist claim and it does not follow from MS. One might surmise that, despite the fact that it gets away with the existence of abstract mathematical objects, respecting the truth of mathematical theorems is enough to characterise MS as a realist theory. I argue that it is not enough and that MS, much like structuralism in general, is quasi-nominalistic in so far as its profound reliance on actuality is concerned and as long as the central questions have to do with the identification of higher-order notions such as relations and functions. Thus the two, fictionalism and MS, certainly are nominalistic in quite different respects. I will thus suggest how to turn MS into a realist one, following the lead from Peirce's account of the nature of mathematics, together with realist interpretation of quantification in second-order modal logic.

MS attempts both (i) to dispense with the nominalistic character of ordinary category-theoretic structuralism and (ii) to avoid the set-theoretic commitments that ensue from the talk about the totality of the universe of mathematical objects. To accomplish these, the absolute universe of set theory is replaced with topoi, each of them a possible world of mathematics. Mathematical domains are indefinitely extendible and relative to these possible worlds in the sense in which mathematical constructions talk about hypothetical constructions. The comprehension schema (see next section) is world-relative, that is, it may procure different mathematical facts in these different possible worlds.

What mathematics actually is thus concerned with is, as Hellman characterises it, "what would *necessarily* be the case *were* the relevant structural conditions fulfilled". Mathematics makes "no actual commitment to objects at all, only to (propositionally) what might be the case" [Hellman 2003, 146]. To understand mathematics, there is thus in principle a lot of rewriting that needs to be done first. To talk, for instance, about infinitely many primes what we are actually saying is, "If there *were* such a thing as a natural number structure, then there *would be* infinitely many primes". This is in essence the core of what Putman suggested back in the 1960s. Mathematics makes no commitment to objects, only to the truth of subjunctive conditionals which express facts about what could or might be the case.

Hellman talks about mathematical necessity in the context in which it does not quite seem to apply, however. Hypothetical (subjunctive) conditionals do not express what the results of necessary, apodictic reasoning are. They express weaker, counterfactual relationships that cannot be interpreted or fully understood in the strictly deductive or naturalistic sense of expressing conditional forms of inference or causal or law-like relationships. They are not about what the axiomatisation of a mathematical system necessarily permits or does not permit to be the case. Rather, what ensues from the fulfilment of relevant structural conditions is that it is *possible* that those structures have certain properties, not that why something necessarily is the case.

There is thus another interesting connection to Peirce's thoughts here. He, too, took mathematics to be of this subjunctive hypothetical kind concerned with what he calls the "*would-bes*" and "*could-bes*".[2] They make up

2. See e.g., [Peirce 1906, CP 4.530, 72]. Also, a decade earlier Peirce wrote: "It is an error to make mathematics consist exclusively in the tracing out of necessary consequences. For the framing of the hypothesis of the two-way spread of imaginary quantity, and the hypothesis of Riemann surfaces, were certainly mathematical achievements. Mathematics is, therefore, the study of the substance of hypotheses, or mental creations, with a view to the drawing of necessary conclusions" [Peirce 1896, IV, 268].

their own ontological category (thirdness) which brings actuality into correspondence with qualities and properties. But such preliminary conceptions of what might or could happen when we enquire about the world, or the inner thought, or the universes of discourse of logic, cannot be identified with point-like structures. It is the interpretation of modalities in respect to which MS is crucially divergent. The proposal faces the philosophical problem precisely in this question of the interpretation of modalities.

3 Cross-world identification and mathematical practice

So where, more precisely speaking, is the problem? Hellman axiomatises modal existence in S5 second-order modal logic. For instance, in MS the logical comprehension schema is:

$$(\text{Comp}) \quad \Box\forall R\forall x_1 x_2 \ldots x_n(R(x_1, x_2, \ldots, x_n) \leftrightarrow \Phi)$$

But no semantics has been provided here to cater for an understanding of how the ensuing logic is supposed to behave. (A related worry which we need to forego here is the question of whether we can understand these axioms and the language of second-order modal logic at all except in set-theoretic terms, in which case their origins would after all be in axiomatic set theory.) Hellman seems to emphasise, just like Peirce would, *mathematical practices* as guiding the decisions to choose and revise axiomatisations of mathematical theories; here such norms would involve looking at the kinds of practices that could contribute to the suitable axioms for the system of second-order modal logic. But whatever these practices may be, they have to contribute to what the semantics of the logical systems would eventually look like. Yet Hellman says next to nothing about what the underlying semantic ideas are that govern the meaning of axioms.

The reason for this omission should be sufficiently clear: otherwise MS would face the grim issue of making sense of the identity between what is the actual and what is the possible with respect to the higher-order entities of mathematics such as relations and functions. Since the current proposals of MS cannot quite make sense of that sort of an identity, Hellman prefers to refrain from quantifying over higher-order entities as that would commit us to somehow identify actual notions with possible ones. Hence we must choose the comprehension schema to apply in each possible world, individually and in isolation, carefully avoiding any cross-identificatory contamination of higher-order notions. As a consequence of such considerations, each possible world comes to constitute its own mathematics.

But why be so suspicious about trans-world identities? Do they echo the Quinean worries about intensional entities, unacceptable not least because we cannot empirically perceive and entertain them? Do the reasons for being mistrustful about the semantic efficiency have something to do with the suspicions of intensional entities in general? Haven't those scruples by now been jettisoned by developments in the semantics of quantified modal logics? According to Hellman, to quantify over relations presupposes the possibility of cross-world identification which according to him would result in a generation of "a universal class of all *possible* objects, and corresponding universal relations among possibilia, directly violating the extendability principle (modally understood, appropriately, as 'Any totality there might be, might be extended')" [Hellman 2005, 554]. For Hellman, this seems to amount to a too "extravagant" ontology that would deprive MS much of its distinctive value. And so he sees it safer to settle for an extensional version of the comprehension scheme.

But the emergence of a universal class of all possible objects does not seem to be the fundamental worry here. It might only be a worry if the semantics for second-order modal logic is taken to be unviable. But why assume that? The reasons are many and there is, for example, a subtle difference as to whether we admit the cross-world identification of higher-order entities to take place among the systems of possible worlds that share the common domains or whether the domains may change from world to world. Extensional comprehension obviously is intended under the common domains assumption and under which the Barcan formulas hold, but that is really only a special case of general modal logics with open domains and in which what the relations instantiate may well vary from world to world.

Even more importantly, it is only through the cross-world comparison of relations and functions, it seems, that we can make sense of what it really means to quantify over second-order entities in modal contexts. And such comparison can only be spelled out in terms of the possibility of identifying the occurrences of these entities from context to context, from one world to another. Cross-world comparisons thus serve as the basis for a realist application of semantic notions that does not take possible worlds as abstract or heuristic models introduced to one's semantic theory merely in order for us to be able to reinterpret our commonplace mathematical notions in some hypothetical but particular and generally arbitrary fashion.

Third, having all possible objects, higher-order ones included, constitutive of the domain of mathematics need not be ontologically extravagant. In fact, an alternative way of looking at the issue is to think of the universes of all possibilia as the cornerstone of, or the continuum for, the kind of realist semantics that the formulations of MS have yet to entertain. In other words, there is a sense in which the extendibility principle is not violated: to understand what it means to quantify over possible relations, for example, presupposes that we have *third-order relations* (say functionals in the category-theoretic sense) at our disposal which serve as the 'meaning functionals' that point out in which contexts (possible worlds) two relations or functions may be identical and in which contexts they may depart from one another. That is, these third-order functions can codify the principles concerning the ways, or habits of action, that mathematicians entertain in their practice of applying second-order mathematical concepts in actual mathematical investigation.

4 Real possibilities to the rescue

These observations, albeit preliminary, suggest that higher-order notions such as relations and functions are the key mathematical notions indispensable in sciences. This effect may be easy to agree with, Harty Field [Field 1989] and his eliminativist programme about numbers notwithstanding. But Hellman's proposal remains eliminativist in certain other senses and thus runs into a trouble when the epistemology of these entities is at issue. Solving the cross-identification issue for individuals just as it would be the matter of the first-order modal logic is not applicable here. Without cross-identification for higher-order notions, we are denying that these relations and mappings are many-world, cross-categorial entities. From the metalogical perspective, we would be prevented from knowing *what* or *which* relations or functions they in fact are. Such a denial means to deprive mathematical knowledge of some of its key facts of the matter. Despite indispensability, MS takes a nominalistic approach to these entities. But if we are to take the indispensability of higher-order notions also to mean our knowledge of them by modes of identification, what lurks around the corner is nominalism with respect to relations and functions.

The fundamental reason for this epistemological deficiency is grounded on the absence of adequate semantic machinery and the resulting leap to eliminativist conclusions concerning the alleged problem of interpreting modalities. Hellman's proposal differs from Peirce's interpretation of modalities in the fundamental sense that Hellman does not take possible objects to be *real* possibilities. Modalities are not constituents of the objective world that accommodates mathematical entities and facts, *even if they would not exist in that world.* Hellman states, unacceptably to my view, that the talk of possible worlds is "heuristic only" and that there are "literally no such things" such as those that merely might have existed [Hellman 2003, 147]. Such claims would follow only if we were predisposed to reject cross-identification of higher-order notions between possible worlds. And the rejection would imply, among other things, that we may never come to know what the meaning of these notions is.

I have indicated merely a possibility for an alternative interpretation. In brief, and to recapitulate the main point, you cannot claim to master your second-order modal logic and deny the reality of possible worlds. Rewriting mathematical propositions into subjunctive forms that assume counterfactual statements is sensible only if there is a factual way of interpreting those counterfactuals. And one cannot claim to have accomplished the latter unless there is a system of possible worlds at play that is not "heuristic only" (à la

Quine or Hellman) but is one that presupposes the possibility of meaningful modes of cross-identifying higher-order mathematical notions. This, in turn, presupposes that the ways of cross-identifying must be taken to depend not only on the entire system of possible worlds in which they may be instantiated but also on the practices and activities by which such concepts are used in mathematics. W.K. Clifford long ago pointed out that every proposition in the sciences, mathematics included, is a *future-oriented rule of conduct*: if such-and-such proposition be true, or such-and-such situation be present, then certain things would happen or certain statable or experiencible consequences may be expected.

5 Conclusions

My task is only to defend the need for a certain semantically explicable metaphysics of modalities according to which to understand modal-structuralist philosophy of mathematics means to understand possible mathematical structures as just as real as the actual ones even though they need not exist. This is the gist of the kind of *scholastic realism* that made an ever-lasting imprint on Peirce's philosophy of mathematics [Moore 2010], [Peirce 2010]. According to Peirce, we must insist on "the reality of some possibilities" [Peirce 1905, CP 5.453]. Hellman would negate; however, both proposals respond to the call for paying closer attention to the practices of mathematics [Pietarinen 2009, 2010]. But if your possibilia is nominalistic, you will never get your full semantics and you will ultimately fail to understand the meanings of the suggested axiomatisations written in scriptures of the second-order modal logic.

A realist reworking of MS removes the redundant concreteness of quasi-nominalist possibilia. A higher-order modal logic interspersed with the semantic machinery of identification of higher-order mathematical entities across possible worlds is guided by what goes on in mathematical practices and not by foundationalism concerning entities or ontology. Since the practices and actions of mathematicians are *fallible* and swim in the continuum of uncertainty and imprecision, they can never be exhausted to constitute a totality: the methods of cross-identifying can never form a completed or closed class of third-order functions.[3]

3. Fallibilism is the view that our current theories of science, including our mathematical theories concerning mathematical facts, may turn out to be false. Reasoning and observation is performed by human beings. Products of science follow from the methods employed in reasoning and observation. There is an element of anthropomorphism in the sciences in the sense that we would never acquire absolute certainty concerning the truth of our best scientific theories. Since Peirce's 'true continuum' adds actuality all that is possible—including all possible mathematical entities, all possible objects, relations, propositions, and facts—and since possibility outweighs actuality, there will be an inevitable uncertainty and vagueness in reality not to be disposed of even by the best theories and methods of sciences. Peirce states fallibil-

Where is the mathematical knowledge located, then? My argument suggests that we do not *know* first and then *do* or *act on* something on the basis of that knowledge. Rather, we act first by various means of experimenting on our initially vague thought-experiments and model constructions, preparing diagrammatic representations for them, and making observations on the outcomes of manipulating such representation. In all of these we are making use of the *iconic forms* of reasoning connected with such representations [Pietarinen 2006, 2014]. Following the acquisition of information that we gain through various practices and habits that govern mathematical conduct we may come to know what the useful mathematical notions are:

> Mathematics might be called an art instead of science were it not that the last achievement that it has in view is an achievement of knowing. [Peirce 1976, III, 523]

Let the final point of support come from the central role that *examples* seem to have in mathematical discovery. Mathematicians tend to be pretty sensitive to the fruitfulness of good examples even if they would not generalise too well. Good mathematical examples seem to have what Peirce termed the "uberty" of scientific hypotheses [Peirce 1913]: their potential to breed future examples that are more likely to generalise in the future and that are thus more plausible in leading into new discoveries. The significance of good examples lies not so much in knowledge of mathematical propositions but in setting imaginative mind into action, orienting it towards future contexts where new examples may facilitate though-experiments and constructions of models by which some axiomatisations of mathematical theories and even knowledge of mathematical propositions can then be sought for.

Bibliography

FIELD, Hartry [1989], *Realism, Mathematics & Modality*, Oxford: Blackwell.

HELLMAN, Geoffrey [1989], *Mathematics without Numbers: Towards a Modal-Structural Interpretation*, Oxford: Clarendon Press.

—— [2003], Does category theory provide a framework for mathematical structuralism?, *Philosophia Mathematica*, 11(2), 129–157, doi:10.1093/philmat/11.2.129.

ism in mathematics as follows: "Mathematical reasoning holds. Why should it not? It relates only to the creations of the mind, concerning which there is no obstacle to our learning whatever is true of them. [...] It is fallible, as everything human is fallible. Twice two may perhaps not be four. But there is no more satisfactory way of assuring ourselves of anything than the mathematical way of assuring ourselves of mathematical theorems. No aid from the science of logic is called for in that field" [Peirce 1902b, CP 2.192].

—— [2005], Structuralism, in: *The Oxford Handbook of Philosophy of Mathematics and Logic*, edited by S. Shapiro, Oxford: Oxford University Press, 536–562.

MOORE, Matthew E. [2010], Scotistic structures, *Cognitio*, 11(1), 79–100.

PEIRCE, Charles S. [1896], On quantity, with special reference to collectional and mathematical infinity, in: *The New Elements of Mathematics by Charles S. Peirce*, edited by C. Eisele, The Hague: Mouton, 1976.

—— [1902a], A detailed classification of the sciences, in: *Collected Papers of Charles Sanders Peirce*, Cambridge, MA: Harvard University Press, 1931–1958.

—— [1902b], General and historical survey of logic: Why study logic?, in: *Collected Papers of Charles Sanders Peirce*, Cambridge, MA: Harvard University Press, 1931–1958.

—— [1905], Issues of pragmaticism, in: *Collected Papers of Charles Sanders Peirce*, Cambridge, MA: Harvard University Press, 1931–1958.

—— [1906], Prolegomena to an apology for pragmaticism, in: *Collected Papers of Charles Sanders Peirce*, Cambridge, MA: Harvard University Press, 1931–1958.

—— [1913], An essay toward improving our reasoning in security and in uberty, in: *The Essential Peirce. Selected Philosophical Writings (1893—1913)*, edited by The Peirce Edition Project, Bloomington: Indiana University Press, vol. 2, 463–474, 1998.

—— [1976], *The New Elements of Mathematics by Charles S. Peirce*, The Hague: Mouton, edited by Eisele, C.

—— [2010], *Philosophy of Mathematics: Selected Writings*, Bloomington: Indiana University Press, edited by M. Moore.

PIETARINEN, Ahti-Veikko [2006], *Signs of Logic: Peircean Themes on the Philosophy of Language, Games, and Communication*, Synthese Library, vol. 329, Dordrecht: Springer, doi:10.1007/1-4020-3729-5.

—— [2009], Pragmaticism as an antifoundationalist philosophy of mathematics, in: *Philosophical Perspectives on Mathematical Practice*, edited by B. Van Kerkhove, R. Desmet, & J. P. Van Bendegem, London: College Publications, 155–182.

—— [2010], Which philosophy of mathematics is Peirce's pragmaticism?, in: *New Essays on Peirce's Mathematical Philosophy*, edited by M. Moore, Chicago; La Salle: Open Court, 59–80.

—— [2014], Is there a general diagram concept?, in: *Thinking with Diagrams*, edited by S. Krämer & C. Ljundberg, Amsterdam; Philadelphia: John Benjamins.

PUTNAM, Hilary [1967], Mathematics without foundations, *The Journal of Philosophy*, 64, 5–22, Reprinted in: P. Benacerraf and H. Putnam (eds.), 1983, *Philosophy of Mathematics: Selected Readings*, Cambridge: Cambridge University Press, 2nd edition, 295–311.